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Specific Equations for One and Two Section Quarter-Wave Matching Networks for Stub-Resistor Loads

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Abstract —**Given a load network consisting of a conductance in parallel with a short-circuited stub, the admittance values of optimum one and two section commensurate transmission line matching networks are derived. These values are expressed in closed form as functions of the bandwidth and ripple level. It is shown that optimum networks have nonzero reflection coefficient minima, as predicted by classical broad-band matching theory.**

I. INTRODUCTION

A CLASSIC PROBLEM in microwave engineering is the broad-band matching of a one-port network consisting of a conductance shunted by a short-circuited stub. A typical example is encountered in the matching of junction circulators. A convenient form of matching network consists of one or more equal length (i.e., commensurate) transmission lines. Several authors have described solutions for the general case having n such lines, the most general result being presented in [1]. A schematic diagram of the network is shown in Fig. 1. It should be noted immediately that this particular form of matching network is not necessarily optimum in having the maximum "gain-bandwidth" product for a given length, but it may be the most convenient for a practical situation. A more optimum network for a similar distributed load network is given in [2], but this is not necessarily so from a practical point of view where realizable impedance levels are a prime consideration.

Papers on matching networks have fallen into two categories. The first consists of sophisticated general solutions such as [1], [2], which have found limited use because of their complexity, not readily appreciated by or comprehensible to a majority of engineers. Included would be the classic papers by Fano [3] and Youla [4]. In the second category are papers which take a more elementary approach, usually involving analysis of one or two section matching networks, leading to results suitable for practical applications, [5], [6]. These methods involve either approximations or solutions to complicated nonlinear simultaneous equations, so that in one sense they are actually more complex than the papers of category 1.

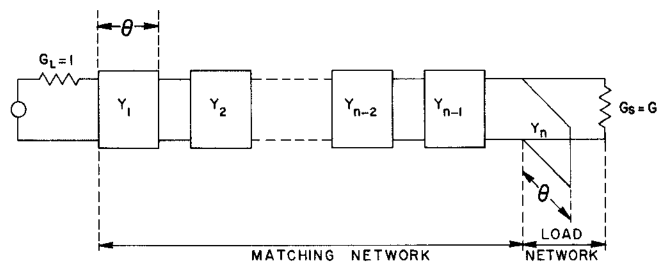

Fig. 1. The general $(n-1)$-section matching network.

One object of this paper is to demonstrate that the classical ("sophisticated") synthesis method of category 1 is actually simpler than the direct ("brute force") method of category 2 when applied to equally simple networks, i.e., with one or two matching elements rather than the general n-element case. Specific equations for the elements of the matching networks result, and computer-derived solutions are not required.

A second objective is to solve the matching problem for the general case where the reflection coefficient minima take on finite values rather than zero. This gives improved

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bandwidth and control over the impedance level within the matching network.

II. Theory

In some previous work the insertion loss function of the network of Fig. 1 has been chosen as a simple Chebyshev function which neglects the pole of attenuation due to the short circuited shunt stub. Although this causes a rather small error for low-ordered networks it is an unnecessary approximation, and here the exact function is used.[1]

The general form of Chebyshev insertion loss function for the network shown in Fig. 1 is [1]

$$L = 1 + K^2 + \epsilon^2 \cdot \left[\frac{(1+\sin\theta_0) T_n\left(\frac{\cos\theta}{\cos\theta_0}\right) - (1-\sin\theta_0) T_{n-2}\left(\frac{\cos\theta}{\cos\theta_0}\right)}{2\sin\theta} \right]^2 \tag{1}$$

where T_n represents the Chebyshev polynomial of the first kind of degree n, and θ is the electrical length of all the transmission lines in the network, i.e.,

$$\theta = \frac{2\pi l}{\lambda}. \tag{2}$$

The parameters defined by (1) are later expressed in terms of the maximum and minimum passband VSWR, and the fractional bandwidth (31)–(33). The $\sin\theta$ term in the denominator takes account of the attenuation poles at $\theta = 0$, $2\pi, \cdots$, etc., and the form of insertion loss for the first harmonic passband centered at $\theta = \pi/2$ is shown in Fig. 2. This insertion loss function may be synthesized for any value of n and the other parameters. In general this involves numerical root-solving of high-degree polynomials, but in the case $n = 2$ and $n = 3$ closed-form solutions may be found, since these cases involve at most the solution of a cubic. The two cases will be taken in turn. The general synthesis method is similar to that given in [7].

III. Case $n = 2$ (Single Section Matching Network)

Here the insertion loss function (1) reduces to

$$L = 1 + K^2 + \epsilon^2 \cdot \left[\frac{(1+\sin\theta_0)\left(\frac{2\cos^2\theta}{\cos^2\theta_0} - 1\right) - (1-\sin\theta_0)}{2\sin\theta} \right]^2. \tag{3}$$

Applying Richards' transformation

$$\tan\theta \to -jt \tag{4}$$

[1]Actually use of the simple Chebyshev function of degree 3 or less gives a result close to the exact solution because only one or two ripples are present, and the function can only be equiripple! The advantage of (1) is in inherently defining the load network and enabling exact closed-form solutions to be obtained.

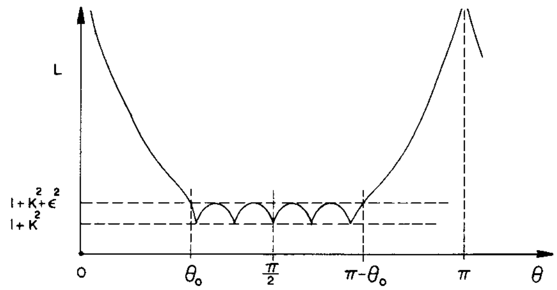

Fig. 2. Insertion loss function ($n = 5$).

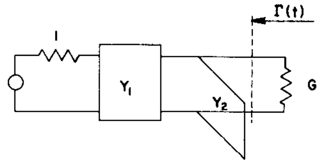

Fig. 3. Single section matching network.

then (3) becomes

$$L = 1 + K^2 + \epsilon^2 \left[\frac{\frac{1+\sin\theta}{\cos^2\theta_0} - (1-t^2)}{jt(1-t)^{1/2}} \right]^2$$

$$= -\frac{(1+K^2)t^2(1-t^2) - \epsilon^2(t^2+\beta)^2}{t^2(1-t^2)} \tag{5}$$

where

$$\beta = \tan^2\theta_0 + \tan\theta_0/\cos\theta_0. \tag{6}$$

Following the Darlington synthesis technique [7], we form the squared modulus reflection coefficient

$$|\Gamma(t)|^2 = \frac{L-1}{L} = -\frac{K^2t^2(1-t^2) - \epsilon^2(t^2+\beta)^2}{(1+K^2)t^2(1-t^2) - \epsilon^2(t^2+\beta)^2}. \tag{7}$$

The numerator of (7) is

$$N(t)N(-t) = (K^2+\epsilon^2)t^4 + 2(\beta\epsilon^2 - K^2/2)t^2 + \beta^2\epsilon^2 \tag{8}$$

$$N(t)N(-t) \equiv at^4 + bt^4 + c \tag{9}$$

$$= (K^2+\epsilon^2)(t^2 - t_1^2)(t^2 - t_1^{*2}) \tag{10}$$

where t_1^* is the complex conjugate of t_1. The denominator of (7) is given by (10) with K^2 replaced by $(1+K^2)$.

If the reflection coefficient is defined as that looking back from the load conductance G, as shown in Fig. 3, then it may be demonstrated that the optimum choice of factorization of (10) is the Hurwitz polynomial having zeros only in the left-half complex plane. The other choice where the zeros are in the right-half plane is a valid solution of the synthesis problem, but gives a nonoptimum result for the matching problem, as discussed in the Appendix. Of course the denominator is forced to be Hurwitz by fundamental considerations.

The Hurwitz factorization of (10) is

$$N(t)=\sqrt{K^2+\epsilon^2}\,(t+t_1)(t+t_1^*)$$
$$=\sqrt{K^2+\epsilon^2}\left[t^2+(t_1+t_1^*)t+|t_1|^2\right]. \quad (11)$$

From (9) and (10) the values of the squared roots are given by

$$t_1^2, t_1^{*2}=\frac{-b\pm j\sqrt{4ac-b^2}}{2a}. \quad (12)$$

Hence

$$|t_1|^2=\sqrt{t_1^2 t_1^{*2}}=\sqrt{\frac{c}{a}} \quad (13)$$

$$(t_1+t_1^*)^2=t_1^2+t_1^{*2}+2|t_1|^2=\frac{-b}{a}+2\sqrt{\frac{c}{a}} \quad (14)$$

from which

$$t_1 t_1^*=\sqrt{\frac{2\sqrt{ac}\;-b}{a}}. \quad (15)$$

This completes the derivation of the Hurwitz numerator polynomial (11), i.e.,

$$N(t)=\sqrt{K^2+\epsilon^2}\left(t^2+\sqrt{\frac{2\sqrt{ac}\;-b}{a}}\,t+\sqrt{\frac{c}{a}}\right) \quad (16)$$

with a, b, and c being the coefficients of (8). The denominator is given by (16) with $1+K^2$ replacing K^2, i.e.,

$$D(t)=\sqrt{1+K^2+\epsilon^2}\left(t^2+\sqrt{\frac{2\sqrt{(a+1)c}\;-b+1}{a+1}}\,t+\sqrt{\frac{c}{a+1}}\right). \quad (17)$$

The reflection coefficient is therefore

$$\Gamma(t)=-\sqrt{\frac{a}{a+1}}\;\frac{t^2+\sqrt{\dfrac{2\sqrt{ac}\;-b}{a}}\,t+\sqrt{\dfrac{c}{a}}}{t^2+\sqrt{\dfrac{2\sqrt{(a+1)c}\;-b+1}{a+1}}\,t+\sqrt{\dfrac{c}{a+1}}}. \quad (18)$$

The negative sign is assigned to $\Gamma(t)$ because at dc ($t=0$) the reflection coefficient corresponds to that of a short circuit, where $\Gamma=-1$.

The output impedance at the plane shown in Fig. 3 is now expressed as

$$Z(t)=\frac{1+\Gamma}{1-\Gamma}=\frac{n_2t^2+n_1t}{d_2t^2+d_1t+d_0} \quad (19)$$

where

$$n_2=\sqrt{a+1}-\sqrt{a} \quad (20)$$
$$d_2=\sqrt{a+1}+\sqrt{a} \quad (21)$$
$$n_1=\sqrt{2\sqrt{(a+1)c}-b+1}-\sqrt{2\sqrt{ac}-b} \quad (22)$$
$$d_1=\sqrt{2\sqrt{(a+1)c}-b+1}+\sqrt{2\sqrt{ac}-b} \quad (23)$$
$$n_0=0 \quad (24)$$

and

$$d_0=2\sqrt{c}. \quad (25)$$

Note that $n_2d_2=1$. This is now to be synthesized to give the network of Fig. 3. Rather than extracting elements using Richards' theorem, etc. (7), it is simpler here to analyze Fig. 1 and use a coefficient comparison method. The transfer matrix of the unit element and stub is

$$\frac{1}{\sqrt{1-t^2}}\begin{bmatrix}1 & t/Y_1\\ Y_1t & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ Y_2/t & 1\end{bmatrix}$$
$$=\frac{1}{\sqrt{1-t^2}}\begin{bmatrix}1+Y_2/Y_1 & t/Y_1\\ Y_1t+Y_2/t & 1\end{bmatrix}. \quad (26)$$

Normalizing the elements of this transfer matrix to the terminations of conductances 1 and G gives the output impedance

$$Z(t)=\frac{D+B}{A+C}\cdot G$$
$$=\frac{1+t/Y_1}{1+Y_2/Y_1+Y_1t+Y_2/t}\cdot G$$
$$=\frac{(G/Y_1)t^2+Gt}{Y_1t^2+(1+Y_2/Y_1)t+Y_2}. \quad (27)$$

Direct comparison of (22) with (27) gives the required final closed-form solution, i.e.,

$$Y_1=\frac{n_1}{n_2} \quad (28)$$
$$Y_2=n_1d_0 \quad (29)$$
$$G=n_1^2 \quad (30)$$

where n_2d_2 has been replaced by unity in (29) and (30).

The above parameters are defined in terms of a, b, and c by (23)–(25), which in turn are given as functions of the original parameters K, ϵ, and θ_0 via (8)–(9) and (6). These are related to the usual parameters of fractional bandwidth w, maximum and minimum VSWR by the relations

$$w=2-4\theta_0/\pi \quad (31)$$

$$S_{\max}=\left(\sqrt{1+K^2+\epsilon^2}+\sqrt{K^2+\epsilon^2}\right)^2 \quad (32)$$

and

$$S_{\min}=\left(\sqrt{1+K^2}+K\right)^2. \tag{33}$$

Note also that $S_{\max}=G/Y_1^2$, as required by a basic relationship at the midband frequency.

IV. CASE $n=3$ (TWO-SECTION MATCHING NETWORK)

Here the general insertion loss function (1) becomes

$$L=1+K^2+\epsilon^2\left[\frac{(1+\sin\theta_0)\left(\dfrac{4\cos^2\theta}{\cos^2\theta_0}-\dfrac{3\cos\theta}{\cos\theta_0}\right)-(1-\sin\theta_0)\dfrac{\cos\theta}{\cos\theta_0}}{2\sin\theta}\right]^2. \tag{34}$$

Applying Richards' transformation (4), after a little algebra the insertion loss becomes

$$L=\frac{(1+K^2)t^2(1-t^2)^2-\dfrac{\epsilon^2}{\cos^2\theta_0}\left[\dfrac{2}{1-\sin\theta_0}-(2+\sin\theta_0)(1-t^2)\right]^2}{t^2(1-t^2)^2}. \tag{35}$$

The squared modulus reflection coefficient is

$$|\Gamma(t)|^2=\frac{L-1}{L}=\frac{N(t)N(-t)}{D(t)D(-t)} \tag{36}$$

where

$$N(t)N(-t)=K^2\left[t^6-t^4\left(2+\frac{\epsilon^2}{K^2}\left(\frac{2+\sin\theta_0}{\cos\theta_0}\right)^2\right)+t^2\left(1-\frac{2\epsilon^2}{K^2}\frac{\sin\theta_0(2+\sin\theta_0)}{(1-\sin\theta_0)^2}\right)-\frac{\epsilon^2}{K^2}\left(\frac{\sin\theta_0(1+\sin\theta_0)}{\cos\theta_0(1-\sin\theta_0)}\right)^2\right]$$

$$\equiv K^2(t^6+pt^4+qt^2+r). \tag{37}$$

As in the previous case ($n=2$), D is the same as N except that K^2 is replaced by $1+K^2$. The numerator polynomial is now expressed as

$$N(t)N(-t)=K^2(t^2-t_1^2)(t^2-t_2^2)(t^2-t^{*2}_2) \quad (\text{for } K\neq 0) \tag{38}$$

where t_1 is real, t_2 complex. In this form of (37) we must have $K\neq 0$. When $K=0$, (38) is replaced by the numerator of (35) with $K=0$, which from (37) results in

$$N(t)N(-t)=\epsilon^2\left[\frac{2+\sin\theta_0}{\cos\theta_0}t^2+\frac{\sin\theta_0}{\cos\theta_0}\frac{1+\sin\theta_0}{1-\sin\theta_0}\right]^2 \quad (\text{for } K=0). \tag{39}$$

This is a perfect square and the Hurwitz factorization is self evident. The zeros lie on the imaginary axis of the t-plane, (this is true also in the case $n=1$ for $K=0$).

The general case (37) or (38) is a cubic in t^2, which may be solved by Cardan's method. Substituting

$$t^2=x-\frac{p}{3} \tag{40}$$

and temporarily ignoring the factor K, (37) becomes the cubic

$$x^3+ax+b \tag{41}$$

where

$$a=q-\frac{p^2}{3} \tag{42}$$

and

$$b=2\left(\frac{p}{3}\right)^3-\frac{pq}{3}+r \tag{43}$$

then the roots of (41) are given by

$$x_1=A+B \tag{44}$$

$$x_2,x_2^*=-\frac{(A+B)}{2}\pm j\frac{\sqrt{3}}{2}\frac{A-B}{2} \tag{45}$$

where

$$A^3,B^3=-\frac{b}{2}\pm\sqrt{\left(\frac{b}{2}\right)^2+\left(\frac{a}{3}\right)^3}. \tag{46}$$

Taking account of (40), the required roots for (38) are

$$t_1^2=A+B-\frac{p}{3} \tag{47}$$

and

$$t_2^2,t_2^{*2}=-\left(\frac{A+B}{2}+\frac{p}{3}\right)\pm j\frac{\sqrt{3}}{2}(A-B)=X_1\pm jX_2. \tag{48}$$

t_2^2, t_2^{*2} are solutions of the quartic (quadratic in t^2)

$$t^4-2X_1t^2+(X_1^2+X_2^2)=0. \tag{49}$$

Occasionally (especially in large bandwidth cases) three real roots may occur, in which case (46) becomes complex, and (48) must be modified accordingly. The Hurwitz fac-

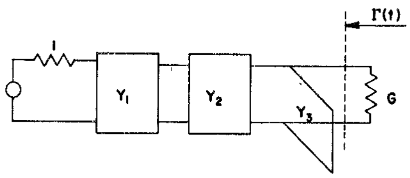

Fig. 4. Two-section matching network.

torization of (38) is given as

$$N(t)=K(t+t_1)\left[t^2+(t_2+t_2^*)t+|t_2|^2\right]$$
$$=K\left[t^3+(t_1+t_2+t_2^*)t^2+\left(|t_2|^2+t_1(t_2+t_2^*)\right)t+t_1|t_2|^2\right]$$
$$=K\left[t^3+\Sigma_1t^2+\Sigma_2t+\Sigma_3\right] \qquad (50)$$

where the real root is given by (47) and the real factors $t_2+t_2^*$ and $|t_2|^2$ required for (50) are obtained from (49).

Following the procedure of (12)–(15) we obtain

$$|t_2|^2=\sqrt{X_1^2+X_2^2} \qquad (51)$$

$$t_2+t_2^*=\sqrt{2\left(X_1+\sqrt{X_1^2+X_2^2}\right)} \qquad (52)$$

This completes the algebra required to determine the numerator polynomial $N(t)$ in closed form. It is only slightly more complicated than the $n=2$ case, involving the simple algebraic relationships (37, 42–43, 46–48, 50–52).

Again, as in the $n=2$ case, the denominator $D(t)$ of $\Gamma(t)$ is obtained from $N(t)$ by the simple expedient of replacing K^2 by $1+K^2$. The reflection coefficient becomes

$$\Gamma(t)=\begin{cases} -\dfrac{K\left(t^3+\Sigma_1t^2+\Sigma_2t+\Sigma_3\right)}{\sqrt{1+K^2}\left(t^3+\Sigma_1't^2+\Sigma_2't+\Sigma_3'\right)} & (\text{for } K\neq 0) \quad (53)\\[2ex] -\dfrac{\epsilon\left(\dfrac{2+\sin\theta_0}{\cos\theta_0}t^2+\dfrac{\sin\theta_0(1+\sin\theta_0)}{\cos\theta_0(1-\sin\theta_0)}\right)}{\left(t^3+\Sigma_1't^2+\Sigma_2't+\Sigma_3'\right)} & (\text{for } K=0). \quad (54)\end{cases}$$

Again, the minus sign is chosen to give $\Gamma(0)=-1$, as required by the presence of the short-circuited shunt stub. The circuit is shown in Fig. 4, and the output impedance is formed as

$$Z(t)=\frac{1+\Gamma(t)}{1-\Gamma(t)}=\frac{n_3t^3+n_2t^2+n_1t}{d_3t^3+d_2t^2+d_1t+d_0}. \qquad (55)$$

The n_0 term in (55) is zero, since comparison of (37) with (53) shows that

$$\Sigma_3=\sqrt{r}\propto\frac{\varepsilon}{K} \quad \text{and} \quad \Sigma_3'=\sqrt{r'}\propto\frac{\varepsilon}{\sqrt{1+K^2}} \qquad (56)$$

so that

$$K\Sigma_3=\sqrt{1+K^2}\,\Sigma_3'. \qquad (57)$$

Proceeding as in the $n=2$ case, the transfer matrix of the reactive portion of Fig. 4 is

$$\frac{1}{1-t^2}\begin{bmatrix}1 & t/Y_1\\ Y_1t & 1\end{bmatrix}\begin{bmatrix}1 & t/Y_2\\ Y_2t & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ Y_3/t & 1\end{bmatrix}$$
$$=\frac{1}{1-t^2}\cdot\begin{bmatrix}1+Y_3(1/Y_1+1/Y_2)+Y_2t^2/Y_1 & (1/Y_1+1/Y_2)t\\ (Y_1+Y_2+Y_1Y_3/Y_2)t+Y_3/t & 1+Y_1t^2/Y_2\end{bmatrix} \qquad (58)$$

giving the output impedance

$$Z(t)=\frac{D+B}{A+C}\cdot G=\frac{(Y_1G/Y_2)t^3+(1/Y_1+1/Y_2)Gt^2+Gt}{(Y_2/Y_1)t^3+(Y_1+Y_2+Y_3Y_1/Y_2)t^2+(1+Y_3(1/Y_1+1/Y_2))t+Y_3}. \qquad (59)$$

Comparing (55) and (59) leads to the expressions

$$\left.\begin{aligned}\frac{Y_2/Y_1}{G}&=\frac{d_3} {n_1} & \frac{1}{Y_1}+\frac{1}{Y_2}&=\frac{n_2}{n_1}\\ \frac{Y_3}{G}&=\frac{d_0}{n_1} & \frac{Y_1}{Y_2}&=\frac{n_3}{n_1}\end{aligned}\right\} \qquad (60)$$

giving the closed-form solutions

$$Y_1=\frac{n_1+n_3}{n_2} \qquad (61)$$

$$Y_2=\frac{n_1Y_1}{n_3} \qquad (62)$$

$$Y_3=n_1d_0 \qquad (63)$$

$$G=n_1^2. \qquad (64)$$

The coefficients $n_1, n_2, \cdots$ are given via (53) and (55). For example,

$$n_3=\sqrt{1+K^2}-K$$
$$n_2=\sqrt{1+K^2}\,\Sigma_1'-K\Sigma_1. \qquad (65)$$

Note also that in obtaining (63)–(64) we have used the fact that $n_3d_3=1$, a similar relationship to the previous case for $n=2$.

To summarize the practical computations, the first step is to determine parameters θ_0, K, and ϵ from (31)–(33). The coefficients p, q, and r are formed in (37), and parameters a, b, A, B, t_1, t_2, t_2^*, X_1, and X_2 are derived from these using (42)–(43) and (46)–(48). The sums and products of root factors Σ_1, Σ_2, and Σ_3 defined in (50) are given by (51)–(52), and similar factors Σ_1', Σ_2', Σ_3' are derived by repeating the entire process with $1+K^2$ replacing K^2. The closed-form expressions for the circuit elements are given in (61)–(64), where the coefficients n_i, d_i are defined by

TABLE I
ONE-SECTION MATCHING NETWORKS

DEGREE N= 2 S(MAX)= 1.2 S(MIN)= 1

W	G	B′	Q	Y1
0.100	54.899	343.593	6.259	8.117
0.150	24.878	102.468	4.119	5.464
0.200	14.371	43.608	3.035	4.153
0.250	9.507	22.568	2.374	3.378
0.300	6.865	13.222	1.926	2.870
0.350	5.272	8.440	1.601	2.515
0.400	4.239	5.737	1.353	2.255
0.450	3.530	4.089	1.158	2.058
0.500	3.023	3.026	1.001	1.905
0.550	2.648	2.307	0.871	1.782
0.600	2.362	1.802	0.763	1.684
0.667	2.077	1.336	0.643	1.579

DEGREE N= 2 S(MAX)= 1.2 S(MIN)= 1.02

W	G	B′	Q	Y1
0.100	52.013	332.465	6.392	7.900
0.150	23.584	99.178	4.205	5.320
0.200	13.634	42.225	3.097	4.045
0.250	9.028	21.863	2.422	3.292
0.300	6.527	12.816	1.964	2.799
0.350	5.019	8.186	1.631	2.454
0.400	4.040	5.567	1.378	2.202
0.450	3.369	3.971	1.179	2.011
0.500	2.889	2.941	1.018	1.862
0.550	2.534	2.243	0.885	1.744
0.600	2.264	1.754	0.774	1.648
0.667	1.995	1.301	0.652	1.547

DEGREE N= 2 S(MAX)= 1.2 S(MIN)= 1.04

W	G	B′	Q	Y1
0.100	48.738	316.180	6.487	7.648
0.150	22.118	94.360	4.266	5.152
0.200	12.801	40.197	3.140	3.919
0.250	8.489	20.827	2.453	3.192
0.300	6.147	12.219	1.988	2.716
0.350	4.735	7.812	1.650	2.384
0.400	3.819	5.318	1.393	2.141
0.450	3.191	3.797	1.190	1.957
0.500	2.742	2.814	1.027	1.814
0.550	2.409	2.149	0.892	1.700
0.600	2.157	1.681	0.779	1.609
0.667	1.905	1.249	0.656	1.512

DEGREE N= 2 S(MAX)= 1.2 S(MIN)= 1.06

W	G	B′	Q	Y1
0.100	45.124	295.209	6.542	7.359
0.150	20.502	88.154	4.300	4.960
0.200	11.885	37.583	3.162	3.776
0.250	7.896	19.492	2.468	3.078
0.300	5.730	11.448	1.998	2.622
0.350	4.425	7.328	1.656	2.304
0.400	3.577	4.994	1.396	2.072
0.450	2.997	3.571	1.192	1.896
0.500	2.582	2.650	1.026	1.760
0.550	2.275	2.026	0.891	1.652
0.600	2.042	1.587	0.777	1.565
0.667	1.809	1.182	0.653	1.473

DEGREE N= 2 S(MAX)= 1.2 S(MIN)= 1.08

W	G	B′	Q	Y1
0.100	41.195	269.883	6.551	7.031
0.150	18.747	80.656	4.302	4.743
0.200	10.890	34.423	3.161	3.615
0.250	7.255	17.876	2.464	2.951
0.300	5.280	10.515	1.991	2.517
0.350	4.090	6.741	1.648	2.215
0.400	3.318	4.602	1.387	1.995
0.450	2.789	3.296	1.182	1.829
0.500	2.411	2.450	1.016	1.701
0.550	2.132	1.877	0.880	1.599
0.600	1.920	1.473	0.767	1.518
0.667	1.708	1.098	0.643	1.432

DEGREE N= 2 S(MAX)= 1.2 S(MIN)= 1.1

W	G	B′	Q	Y1
0.100	36.949	240.386	6.506	6.659
0.150	16.851	71.919	4.268	4.497
0.200	9.818	30.739	3.131	3.432
0.250	6.563	15.991	2.437	2.806
0.300	4.795	9.424	1.965	2.399
0.350	3.730	6.055	1.623	2.116
0.400	3.039	4.143	1.363	1.910
0.450	2.566	2.974	1.159	1.755
0.500	2.228	2.216	0.994	1.635
0.550	1.979	1.701	0.859	1.541
0.600	1.790	1.337	0.747	1.465
0.667	1.601	1.000	0.625	1.386

comparison of (53) and (55), as in (65). If $K=0$, corresponding to $S_{\min}=1$, then (54) is used rather than (53), and the denominator is derived from (37) by setting $K=1$.

The validity of (61)–(64) may be tested by recognizing the following relationship between Y_1, Y_2, and G, obtained from the input admittance at the center frequency:

$$S_{\min}=\frac{Y_2^2}{Y_1^2 G}. \tag{66}$$

V. RESULTS

Results for one and two section matching networks are given in Tables I and II. Only a few examples are given since they are easily derived from the closed-form results presented in the text. In order to enable comparison to be made with previous results, e.g., [6], the stub admittance Y_{n+1} is replaced by its equivalent susceptance slope parameter B', where

$$B'=\frac{\pi Y_{n+1}}{4}. \tag{67}$$

Similarly the Q factor of the load network is listed, where

$$Q=B'/G \tag{68}$$

The object of this work is to maximize the bandwidth for a given Q, or alternatively maximize Q for a given bandwidth. Inspection of the tables shows that there is an optimum value of $S_{\min}$ which maximizes Q for a given bandwidth, as predicted by basic matching theory [3], [4]. However the value of Q varies rather slowly with $S_{\min}$, i.e., the optimum region is rather flat. The optimum value of $S_{\min}$ is roughly halfway between $S_{\max}$ and unity.

Since we have derived a closed-form expression for Q in terms of the network parameters, including $S_{\min}$, in theory it should be possible to differentiate this expression with respect to $S_{\min}$ to determine an expression for its true optimum value. However, the differentiation is extremely tedious, and the turning-point relationship is a high-order polynomial in $S_{\min}$. Since, as stated above, many results are easily tabulated and the optimum region is so flat, formal solution of the optimum appears to be an unnecessary exercise.

Of equal interest is the fairly wide range of impedance levels which may be obtained by changing $S_{\min}$, resulting in some degree of flexibility in choice of dielectric materials for realizing the impedances of the matching transmission lines. Examination of the tabulated results indicates that

TABLE II
TWO-SECTION MATCHING NETWORKS

DEGREE N= 3 S(MAX)= 1.15 S(MIN)= 1.08

W	G	B′	Q	Y1	Y2
0.200	580.786	2253.269	3.880	3.881	97.189
0.250	240.220	736.770	3.067	3.144	50.639
0.300	117.341	295.436	2.518	2.660	29.945
0.350	64.382	136.404	2.119	2.321	19.351
0.400	38.519	69.843	1.813	2.071	13.360
0.450	24.657	38.714	1.570	1.882	9.713
0.500	16.672	22.852	1.371	1.735	7.362
0.550	11.801	14.198	1.203	1.618	5.777
0.600	8.687	9.205	1.060	1.524	4.668
0.667	6.083	5.458	0.897	1.425	3.653
0.700	5.193	4.288	0.826	1.384	3.279
0.750	4.187	3.052	0.729	1.332	2.832
0.800	3.458	2.224	0.643	1.288	2.489
0.850	2.918	1.653	0.566	1.251	2.222
0.900	2.511	1.250	0.498	1.220	2.010
0.950	2.200	0.961	0.437	1.194	1.840
1.000	1.957	0.748	0.382	1.171	1.703

DEGREE N= 3 S(MAX)= 1.15 S(MIN)= 1.1

W	G	B′	Q	Y1	Y2
0.200	452.332	1742.610	3.853	3.572	79.668
0.250	187.508	570.432	3.042	2.900	41.652
0.300	91.859	229.069	2.494	2.460	24.733
0.350	50.587	105.957	2.095	2.153	16.060
0.400	30.402	54.376	1.789	1.928	11.150
0.450	19.567	30.223	1.545	1.758	8.157
0.500	13.315	17.897	1.344	1.627	6.225
0.550	9.494	11.160	1.175	1.523	4.921
0.600	7.047	7.266	1.031	1.440	4.008
0.667	4.996	4.334	0.868	1.353	3.171
0.700	4.293	3.417	0.796	1.317	2.862
0.750	3.497	2.445	0.699	1.272	2.494
0.800	2.920	1.791	0.613	1.234	2.212
0.850	2.492	1.338	0.537	1.203	1.991
0.900	2.168	1.018	0.470	1.176	1.816
0.950	1.920	0.787	0.410	1.154	1.677
1.000	1.727	0.616	0.356	1.135	1.564

DEGREE N= 3 S(MAX)= 1.15 S(MIN)= 1.12

W	G	B′	Q	Y1	Y2
0.200	309.011	1152.644	3.730	3.173	59.027
0.250	128.630	378.096	2.939	2.587	31.049
0.300	63.360	152.247	2.403	2.205	18.573
0.350	35.134	70.666	2.011	1.939	12.165
0.400	21.296	36.419	1.710	1.746	8.529
0.450	13.846	20.345	1.469	1.602	6.308
0.500	9.535	12.120	1.271	1.491	4.872
0.550	6.891	7.609	1.104	1.404	3.901
0.600	5.192	4.991	0.961	1.335	3.220
0.667	3.763	3.010	0.800	1.264	2.596
0.700	3.272	2.387	0.730	1.236	2.366
0.750	2.714	1.724	0.635	1.199	2.091
0.800	2.308	1.274	0.552	1.170	1.881
0.850	2.006	0.961	0.479	1.145	1.717
0.900	1.778	0.738	0.415	1.125	1.587
0.950	1.602	0.575	0.359	1.108	1.484
1.000	1.466	0.454	0.310	1.093	1.401

DEGREE N= 3 S(MAX)= 1.15 S(MIN)= 1.14

W	G	B′	Q	Y1	Y2
0.200	138.408	458.742	3.314	2.528	31.758
0.250	58.354	151.441	2.595	2.084	16.998
0.300	29.231	61.496	2.104	1.799	10.384
0.350	16.559	28.850	1.742	1.604	6.968
0.400	10.302	15.064	1.462	1.465	5.020
0.450	6.908	8.546	1.237	1.363	3.825
0.500	4.928	5.182	1.051	1.287	3.050
0.550	3.704	3.317	0.896	1.229	2.525
0.600	2.911	2.222	0.763	1.184	2.157
0.667	2.238	1.381	0.617	1.139	1.820
0.700	2.005	1.111	0.554	1.122	1.696
0.750	1.739	0.820	0.472	1.100	1.549
0.800	1.545	0.620	0.401	1.083	1.437
0.850	1.400	0.477	0.341	1.069	1.351
0.900	1.290	0.374	0.290	1.058	1.283
0.950	1.523	0.410	0.270	1.084	1.428
1.000	1.738	0.446	0.257	1.105	1.556

DEGREE N= 3 S(MAX)= 1.15 S(MIN)= 1

W	G	B′	Q	Y1	Y2
0.200	969.349	3486.515	3.597	4.732	147.312
0.250	399.553	1138.049	2.848	3.818	76.309
0.300	194.299	455.324	2.343	3.214	44.805
0.350	106.011	209.635	1.977	2.789	28.712
0.400	62.989	106.970	1.698	2.474	19.635
0.450	39.988	59.049	1.477	2.233	14.123
0.500	26.777	34.687	1.295	2.044	10.579
0.550	18.741	21.430	1.143	1.893	8.196
0.600	13.620	13.805	1.014	1.770	6.533
0.667	9.356	8.106	0.866	1.639	5.013
0.700	7.902	6.336	0.802	1.584	4.454
0.750	6.265	4.472	0.714	1.514	3.789
0.800	5.083	3.229	0.635	1.454	3.277
0.850	4.211	2.378	0.565	1.403	2.878
0.900	3.555	1.782	0.501	1.359	2.562
0.950	3.054	1.356	0.444	1.321	2.309
1.000	2.665	1.045	0.392	1.289	2.104

DEGREE N= 3 S(MAX)= 1.15 S(MIN)= 1.02

W	G	B′	Q	Y1	Y2
0.200	893.332	3313.304	3.709	4.559	137.630
0.250	368.379	1081.745	2.936	3.681	71.355
0.300	179.240	432.918	2.415	3.102	41.940
0.350	97.862	199.388	2.037	2.693	26.909
0.400	58.196	101.785	1.749	2.392	18.428
0.450	36.983	56.215	1.520	2.161	13.275
0.500	24.794	33.042	1.333	1.981	9.961
0.550	17.378	20.428	1.176	1.836	7.731
0.600	12.649	13.170	1.041	1.719	6.175
0.667	8.709	7.742	0.889	1.594	4.752
0.700	7.366	6.055	0.822	1.542	4.228
0.750	5.852	4.279	0.731	1.475	3.604
0.800	4.758	3.093	0.650	1.418	3.125
0.850	3.951	2.280	0.577	1.370	2.751
0.900	3.344	1.711	0.512	1.329	2.455
0.950	2.880	1.303	0.453	1.294	2.217
1.000	2.520	1.006	0.399	1.263	2.025

DEGREE N= 3 S(MAX)= 1.15 S(MIN)= 1.04

W	G	B′	Q	Y1	Y2
0.200	801.796	3043.685	3.796	4.363	125.987
0.250	330.850	994.045	3.005	3.525	65.394
0.300	161.116	397.989	2.470	2.974	38.491
0.350	88.060	183.398	2.083	2.585	24.738
0.400	52.436	93.683	1.787	2.298	16.973
0.450	33.374	51.781	1.552	2.080	12.253
0.500	22.416	30.463	1.359	1.909	9.216
0.550	15.744	18.854	1.198	1.772	7.171
0.600	11.488	12.170	1.059	1.662	5.743
0.667	7.939	7.168	0.903	1.544	4.437
0.700	6.728	5.612	0.834	1.495	3.956
0.750	5.362	3.972	0.741	1.432	3.383
0.800	4.375	2.876	0.657	1.379	2.942
0.850	3.646	2.124	0.583	1.334	2.598
0.900	3.098	1.597	0.515	1.296	2.326
0.950	2.678	1.219	0.455	1.263	2.108
1.000	2.353	0.943	0.401	1.235	1.932

DEGREE N= 3 S(MAX)= 1.15 S(MIN)= 1.06

W	G	B′	Q	Y1	Y2
0.200	697.205	2688.010	3.855	4.139	112.522
0.250	287.967	878.302	3.050	3.348	58.498
0.300	140.406	351.866	2.506	2.828	34.498
0.350	76.861	162.270	2.111	2.462	22.222
0.400	45.854	82.970	1.809	2.193	15.286
0.450	29.252	45.912	1.570	1.988	11.067
0.500	19.700	27.047	1.373	1.827	8.351
0.550	13.880	16.766	1.208	1.700	6.521
0.600	10.163	10.841	1.067	1.597	5.242
0.667	7.062	6.402	0.907	1.488	4.071
0.700	6.002	5.020	0.836	1.443	3.640
0.750	4.806	3.561	0.741	1.385	3.126
0.800	3.941	2.585	0.656	1.336	2.731
0.850	3.302	1.914	0.580	1.295	2.423
0.900	2.820	1.443	0.512	1.260	2.178
0.950	2.451	1.104	0.450	1.230	1.983
1.000	2.165	0.856	0.396	1.205	1.825

the case corresponding to $S_{min}=1$ leads to an upper bound on the absolute values of G and B'. There is a useful reduction as S_{min} increases beyond the values of S_{min} corresponding to the optimum bandwidth. This is of considerable value in the design of junction circulators, in that it is often difficult to realize large values of G and B'. Since the loaded Q of the circuit is almost independent of S_{min}, it allows some uncertainly in their absolute values to be accomodated, provided that their ratio satisfies the required Q of the load circuit.

Practical circulators have often been designed with a final result corresponding to $S_{min}\neq 1$, but possibly without theoretical comprehension of the network problem.

V. Conclusions

The element values of one and two section quarter-wave matching networks for a stub-resistor load network have been derived in closed form as functions of bandwidth and ripple level, with S_{min} different from unity. This represents an advance over previous results for these simple matching networks, where the solution has been given only in implicit form for the single section (degree $n=2$) case [8].

The higher degree ($n=3$) case has particular application in the design of quarter-wave-coupled junction circulators of octave bandwidth or more.

The ability to vary S_{min} leads to significantly useful variation of impedance levels within the structure in addition to optimization of bandwidth. It explains previous empirical results observed in practical junction circulators.

Appendix

The general matching problem is shown in Fig. 5, where M is the matching network. In deriving the network M we have formed the reflection coefficient $\Gamma_{22}(t)$ looking back from the load admittance. By writing down the unitary condition on the lossless scattering matrix it can be shown [1] that if

$$\Gamma_{22}(t)=\frac{N_{11}(t)}{D(t)} \tag{69}$$

then

$$\Gamma_{11}(t)=\frac{N_{11}(-t)}{D(t)}. \tag{70}$$

Hence if a set of zeros is chosen for $\Gamma_{22}(t)$, then the complementary set appears in $\Gamma_{11}(t)$.

When the network M is a cascade of commensurate transmission lines, and if $\Gamma_{22}(t)$ has zeros all in the left-hand complex plane, then synthesis with the opposite choice of zeros corresponds to a Kuroda transformation, with the shunt stub transformed through the first network to be adjacent to the other resistive termination. In the cases considered in this paper there is no difference between the two choices if $S_{min}=1$ or $K=0$, because then the reflection coefficient zeros (by definition) all lie on the imaginary axis of the complex plane. It is interesting that for this special case the Kuroda transformation results in an identical network.

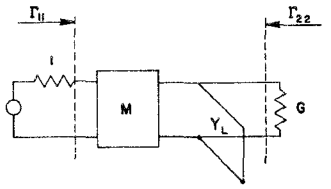

Fig. 5. General matching problem.

The reason why $\Gamma_{22}(t)$ should have a Hurwitz numerator is given by consideration of basic gain-bandwidth theory. In the case of the load network of Fig. 5 the following integral must be satisfied [2]–[4]:

$$\int_0^\infty \frac{1}{\Omega^2}\ln\frac{1}{|\Gamma_{22}|^2}\,d\Omega=\frac{2\pi G}{Y_L}-\sum_i\frac{1}{t_i} \tag{71}$$

where Ω is the real frequency axis of the t-plane, and the t_i denote the right-half plane zeros of $\Gamma_{22}(t)$. The summation indicated in (71) is a positive real quantity since these zeros are real or occur in complex-conjugate pairs. The integral on the left-hand side of (71) is invariant to choice of zero distribution between the left and right half-planes because it involves only the modulus of Γ_{22}. In order to maximize the $Q(=(\pi/4)(Y_L/G))$ it is necessary to minimize G/Y_L, i.e., to make the reciprocal root summation in (71) zero. Hence the optimum choice to maximize Q for a given bandwidth requires $\Gamma_{22}(t)$ to be chosen with zeros only in the left-hand complex t-plane.

This condition has been investigated in the present matching problem by choosing the zeros incorrectly in the right half-plane. It was determined that this caused large reductions in Q values, and significantly the maximum values of Q occurred for $S_{min}=1$, i.e., the Q values decreased montonically as S_{min} increased for a fixed values of S_{max} and bandwidth.

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Noise in Broad-Band GaAs MESFET Amplifiers with Parallel Feedback

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Abstract—**The influence of the circuit elements of a single-ended feedback amplifier module on noise figure and gain, as well as on input and output reflection coefficients is discussed. Theoretical results are supported by tests performed on a five-stage single-ended amplifier. The unit exhibits 41.5±0.8 dB of small-signal gain and a maximum noise figure of 4.0 dB between 2.4 and 8.0 GHz. Maximum reflection coefficients of 1.7:1 for the input and 1.5:1 for the output terminal were measured. The unit's overall circuit dimensions are 25×3.6 mm.**

I. Introduction

RECENT ADVANCES in the performance of single-ended microwave feedback amplifiers have resulted in a device that promises to challenge the conventional type amplifier in many applications. This is especially the case whenever compact size and low cost are a factor in broad-band microwave amplification [1], [2]. In addition to multi-octave bandwidths and low-reflection coefficients, the feedback amplifier shows great potential for low-noise applications. This is true in spite of the thermal noise injected by the feedback resistor.

The influence of reactive feedback on the noise figure of microwave amplifiers has been studied by several researchers [3]–[5]. It has also been pointed out that reactive feedback reduces the minimum noise figure of microwave amplifiers, a fact that has been known to designers of VHF amplifiers for several decades [6].

This paper addresses the noise in broad-band microwave amplifiers with parallel feedback. Formulas for the equivalent noise parameters and noise figure of such amplifiers are presented. They take into account the thermal noise agitation of the resistor in the feedback loop. Based on these theoretical solutions, the influence of the circuit elements on noise figure, gain, and reflection coefficients of a practical amplifier are discussed. Attention is focused on

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